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ZARISKI EXTENSIONS AND BIREGULAR RINGS

BY

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ABSTRACT

In this paper we study the structure of Zariski central rings with regular center i.p. biregular rings, and we obtain structure theorems for algebras which are finitely generated over their regular center, etc. Characterizations of certain classes of rings are being obtained by using localization at prime ideals and local-global theorems.

Introduction

In Section 1 of this paper we characterize biregular rings as being those fully idempotent rings which are Zariski central rings in the sense of [17]. Consequently, biregular rings are exactly those rings for which central localization at maximal ideals yields simple rings, in other words, biregular rings may also be characterized as being the Zariski central rings for which symmetric localization at maximal ideals (in the sense of [16]) yields simple rings of quotients. It is interesting to note the parallels between these results and the characterization of regular rings in the class of P.I.-rings in [2].

In the class of fully left bounded Zariski central rings, being fully left idempotent is equivalent to being fully idempotent, biregular, a left V-ring. This straigthens out an error in [15].

In [9], G. Michler and O. Villamayor studied regularity of P.I.-rings which are finitely generated as algebras over the center. We give a more general characterisation of biregular rings which are just finitely generated algebras over the center; our result applied to the P.I.-rings studied in [9] yields G. Michler and O. Villamayor's result that for a P.I.-ring which is a finitely generated algebra over its center, being regular is equivalent to being biregular. We also characterize which left Goldie rings are fully left bounded biregular and show that these are exactly the semisimple Artinian rings.

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In Section 2 we study prime ideals and localization at a prime ideal. Our main result here is that the rings of quotients associated to the prime ideal P by localizing resp. at J. Lambek and G. Michler's kernel functor κ_P and at F. Van Oystaeyen's symmetric kernel functor κ_{R-P}, have isomorphic centers. Since regularity of the center is one of the main tools in this theory the result mentioned has many applications for the study of (bi)regular rings. Indeed, whereas exactness of the symmetric localization functor Q_{R-P} associated to P is easily checked in the situations we have here, exactness of Q_P is linked to the left Ore condition with respect to $G(p) = \{r \in R, rx \in P \Rightarrow x \in P\}$. For a left Noetherian biregular ring it is true that, for every prime ideal P of R, R satisfies the left Ore conditions with respect to $G(P)$. In the absence of the Noetherian condition we show that a fully left bounded Zariski central ring with regular center also satisfies the (left) Ore conditions at prime ideals and, moreover, both localizations mentioned above coincide in this case. The latter class does not consist of biregular rings alone; the result applies to integral extensions of the center and rings which are module finite over their center. In the latter case we are able to recover and extend a result of E. Armendariz [1].

0. Preliminaries

All rings are associative with unit. If R is a ring then R-mod stands for the Grothendieck category of left R-modules. A two-sided ideal will be referred to as an ideal. A ring R with center C is said to be a *global Zariski extension* of C (or Zariski central) if for every ideal H of R we have that $\operatorname{rad} H = \operatorname{rad} R(H \cap C)$ where rad stands for the prime radical, cf. [17].

R is *fully* (left) *idempotent* if $L^2 = L$ for every (left) ideal L of R. If R is fully idempotent then all ideals of R are semiprime and $IJ = I \cap J$ for ideals I, J of R.

R is *regular* if and only if every principal left ideal of R is generated by an idempotent element, equivalently, if and only if for each $r \in R$ there is an $x \in R$ such that $r = rxr$.

R is *biregular* if each of its principal ideals is generated by a central idempotent. A *left V-ring* is a ring R such that the (left) simple R-modules are (left) injective, equivalently, if every left ideal is an intersection of maximal left ideals.

In the commutative case all definitions given here are equivalent to one another. In general we have that regular rings, biregular rings as well as left V-rings, are fully left idempotent rings and, moreover, the center of a fully idempotent ring is a regular ring. For some results on central localization of these rings we refer to [3].

1. Biregular rings

LEMMA 1.1. *Let R be a ring with regular center C, then:*

(1) *$RI \cap C = I$ for every ideal I of C.*

(2) *For each $m \in \Omega(C)$ (Ω will always denote the set of maximal ideals of the ring considered) there exists an $M \in \Omega(R)$ such that $M \cap C = m$.*

PROOF. (1) If $a \in RI \cap C$ then $a = \sum_{i=1}^{n} r_i a_i$, $r_i \in R$, $a_i \in I$. Since $Ca_1 + \cdots + Ca_n$ is a finitely generated ideal of C it is generated by an idempotent element $e \in I$. Hence $a = re$ for some $r \in R$, i.e. $a = ae \in I$.

(2) By (1): $Rm \neq R$, hence $Rm \subset M$ for some $M \in \Omega(R)$ and obviously $M \cap C = m$. □

THEOREM 1.2. *Let R be a ring with center C. The following properties are equivalent:*

(1) *C is regular and for each $M \in \Omega(R)$, $R(M \cap C) = M$.*

(2) *R is fully left idempotent and a global Zariski extension of C (i.e. Zariski central in the sense of [17]).*

(3) *R is fully idempotent and a global Zariski extension of C.*

(4) *R is a biregular ring.*

PROOF. (1) $\Rightarrow$ (2). Take $m \in \Omega(C)$. By Lemma 1.1. we may find $M \in \Omega(R)$ such that $M \cap C = m$ and $Rm = M$ by (1).

From [3] we have: $Q_{C-m}(R) = R/Rm$, where Q_{C-m} denotes central localization at m. Therefore $Q_{C-m}(R)$ is a simple ring hence fully left idempotent. Again from [3] it then follows that R is fully left idempotent. For Zariski centrality of R we have to check whether for each ideal H of R, $\operatorname{rad} H = \operatorname{rad}(H \cap C)$ where rad denotes the prime radical. First, note that prime ideals of R are maximal. Indeed, if $P \in \operatorname{Spec} R$ then $P \cap C \in \Omega(C)$ and there is an $M \in \Omega(R)$ such that $M \cap C = P \cap C$. However, then $M = R(P \cap C) \subset P$ yields $M = P$. Consider now $M \in \Omega(R)$ and let H be an ideal of R such that $H \cap C \subset M \cap C$. If $H \not\subset M$ then $M + H = R$, i.e. $1 = a + b$ for some $a \in M$, $b \in H$. Since $M = R(M \cap C) = Rm$, and $Rm = \kappa_{C-m}(R)$ is the torsion ideal with respect to the central kernel functor associated to the multiplicative set $C - m$, it follows that $sa = 0$ for some $s \in C - m$. Hence $s = sb \in H \cap C$ but this contradicts $s \notin m$, thus $H \subset M$. This fact together with the lying over property for prime ideals, yields that R is Zariski central and even such that every $m \in \Omega(C)$ is contracted from some $M \in \Omega(R)$.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). For any ideal H of R, $\operatorname{rad} H = \operatorname{rad} R(H \cap C)$.

However, R being fully left idempotent, every ideal of R is semiprime and thus $H = R(H \cap C)$. Choose now H to be a principal ideal of R, i.e. $H = RaR$, $a \in R$. Then $a = r_1a_1 + \cdots + r_na_n$ with $r_i \in R$, $a_i \in H \cap C$. Then ideal $Ca_1 + \cdots + Ca_n$ is finitely generated in the regular ring C thus it is generated by an idempotent, e say. Consequently $a \in \mathrm{Re}$ and thus $H \subset Re$ or $H = Re$.

(4) ⇒ (1). C is regular. For any ideal H of R we have that $R(H \cap C) = H$ because if $a \in H$ then $RaR = Re$ for some idempotent element $e \in H \cap C$. □

This characterization of biregular rings in the class of fully idempotent rings now yields the following strengthening of results of [3].

Proposition 1.3. *Let R be a ring with center C. The following statements are equivalent*:

(1) *R is biregular.*

(2) *$Q_{C-m}(R)$ is a simple ring for each maximal ideal m of C.*

(3) *$Q_{R-M}(R)$ is simple for each $M \in \Omega(R)$ and R is a global Zariski extension of C.*

Proof. (1) ⇒ (2). See the implication (1) ⇒ (2) in Theorem 1.2.

(2) ⇒ (1). If $m \in \Omega(C)$ then $Q_{C-m}(R)$ is simple hence fully idempotent and thus C is regular. Consequently $Q_{C-m}(R) = R/Rm$ for each $m \in \Omega(C)$, thus Rm is a maximal ideal of R. By Theorem 1.2. This yields that R is biregular.

(1) ⇒ (3). By Theorem 1.2, (1) implies that R is a global Zariski extension of C and therefore, if $M \in \Omega(R)$, then $Q_{R-M}(R) \cong Q_{C-m}(R)$ where $m = M \cap C$, cf. [17]; note that Q_{R-M} is the symmetric localization functor at the m-system $R - M$ introduced in [11].

Clearly this yields that $Q_{R-M}(R)$ is simple.

(3) ⇒ (1). Take H to be any ideal of R. For each $M \in \Omega(R)$, $Q_{R-M}(R)$ is simple thus: $Q_{R-M}(R)j_M(H)Q_{R-M}(R)j_M(H) = Q_{R-M}(R)j_M(H)$, where j_M is the canonical ring homomorphism $j_M : R \to Q_{R-M}(R)$ with $\operatorname{Ker} j_M = \kappa_{R-M}(R)$. Since R is a global Zariski extension of C, ideals of R extend to ideals of $Q_{R-M}(R)$, cf. [17], therefore $Q_{R-M}(R)j_M(H^2) = Q_{R-M}(R)j_M(H)$. Pick $a \in H$. Since the foregoing states exactly that H/H^2 is a κ_{R-M}-torsion module we may find an ideal I_M of R, $I_M \not\subset M$ and such that $I_Ma \subset H^2$. The ideal $I = \Sigma_{M \in \Omega(R)} I_M$ is not contained in any $M \in \Omega(R)$, thus $I = R$. However $Ia \subset H^2$, i.e. $a \in H^2$.

Consequently $H = H^2$, and Theorem 1.2 finishes the proof. □

Remarks. (1) If for each $M \in \Omega(R)$, κ_{R-M} is an idempotent kernel functor on R-mod such that ideals of R extend to ideals of $Q_{R-M}(R)$ and if each $Q_{R-M}(R)$ is fully idempotent, then R is fully idempotent.

(2) For details on localization at kernel functors, in particular those associated to prime ideals, we refer to [7], [8], [16].

In [15] the authors claim to have established a class of rings not consisting of fully left bounded rings, and such that regularity is equivalent to being a V-ring (cf. [15] prop. 9). This is not correct and we give the correction in the following proposition. Note that, although R is supposed to be fully left bounded here, it is not necessarily taken to be left Noetherian!

PROPOSITION 1.4. *Let R be a fully left bounded ring which is a global Zariski extension of its center C. The following conditions are equivalent:*

(1) *R is fully left (right) idempotent.*

(2) *R is fully idempotent.*

(3) *R is biregular.*

(4) *R is regular.*

(5) *R is a left (right) V-ring.*

PROOF. (1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). Theorem 1.2.

(3) $\Rightarrow$ (4). By biregularity of R, Spec $R = \Omega(R)$. Hence for each $P \in$ Spec R, R/P is a simple ring and also left bounded, i.e. simple Artinian. Thus each prime factor ring of R is regular and since R is fully idempotent we may apply a result of [6] stating that in this case R is regular too.

(3) $\Rightarrow$ (5). As before it follows that each prime factor ring is simple Artinian hence a left V-ring, and R is fully idempotent. Then from [5] it follows that R is a left V-ring.

(4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) are well known. □

REMARK. A closer look at the proof of (1) $\Rightarrow$ (2) in Proposition 1.3 shows that R is a fully left bounded biregular ring if and only if $Q_{C-m}(R)$ is a simple Artinian ring for every $m \in \Omega(C)$.

From Proposition 1.4 we deduce that a P.I.-ring (cf. [13]) is a biregular ring if and only if it is regular and a global Zariski extension of its center. One should compare this to theorem 7 in [3], characterizing those P.I. rings whose central localizations at maximal ideal of the center are simple!

There do exist P.I. rings which are regular but not biregular, e.g. the ring of all sequences of 2×2 matrices in a field, which are eventually diagonal, cf. [3].

Moreover, for P.I. rings the properties of being fully idempotent, fully left idempotent, regular, and left V-ring are equivalent, cf. [6]. We now turn to rings which are algebra-finite over their center.

PROPOSITION 1.5. *Let R be a ring with regular center and let R be a finitely generated C-algebra. Then the center of $Q_{C-m}(R) = R/Rm$ is isomorphic to C/m for each $m \in \Omega(C)$.*

PROOF. Since C is regular: $\kappa_{C-m}(R) = Rm$, $Rm \cap C = m$ and $Q_{C-m}(R) = R/Rm$. Let $j: R \to R/R_m$ be the canonical epimorphism and let $j(r) \in Z(R/Rm)$, i.e. $rx - xr \in Rm$ for all $x \in R$. Let $\{g_1, \cdots, g_n\}$ generate R as a C-algebra, then for each g_i there exists an $s_i \in C - m$ such that $s_i(rg_i - g_ir) = 0$. Put $s = s_1 \cdots s_n \in C - m$. In order to have that $s(rx - xr) = 0$ for every $x \in R$ it will be sufficient to check that $s(rg_{i_1} \cdots g_{i_k} - g_{i_1} \cdots g_{i_k} r) = 0$ for any monomial $g_{i_1} \cdots g_{i_k}$. However the latter follows from:

$$rg_{i_1} \cdots g_{i_k} - g_{i_1} \cdots g_{i_k} r = (rg_{i_1} - g_{i_1} r) g_{i_2} \cdots g_{i_k} + g_{i_1}(rg_{i_2} \cdots g_{i_k} - g_{i_2} \cdots g_{i_k} r).$$

Thus $s(rx - xr) = 0$ for any $x \in R$, i.e. $sr \in C$. Now $s \in C - m$ entails that $j(s)$ is invertible in $j(C) \cong C/m$. So $j(s)h(r) \in j(C)$ yields $j(r) \in j(C) \cong C/m$. □

COROLLARY 1.6. *Let R be a ring which is a finitely generated algebra over its center C, then R is biregular if and only if $Q_{C-m}(R)$ is biregular for each $m \in \Omega(C)$.*

PROOF. Suppose $Q_{C-m}(R)$ is biregular for each $m \in \Omega(C)$, then $Q_{C-m}(R)$ is fully idempotent too and hence R is fully idempotent, i.e. the center C is regular. By the foregoing proposition, the center of $Q_{C-m}(R)$ is a field but non-trivial ideals of biregular rings intersect the center non-trivially, thus $Q_{C-m}(R)$ is simple for each $m \in \Omega(C)$. As $Q_{C-m}(R) = R/mR$, $Rm \in \Omega(R)$ and Theorem 1.2 yields that R is biregular. The converse implication is trivial. □

Combining this corollary with Propositions 1.4 and 1.5 we have generalized a result of Michler–Villamayer [9].

COROLLARY 1.7. *If R is a* P.I. *ring which is finitely generated as an algebra over its center C then R is regular if and only if R is biregular.*

PROOF. If R is regular then C is regular and by Proposition 1.5 the center of R/mR is a field, for each $m \in \Omega(C)$. On the other hand, regularity of R yields that R/mR is a semiprime ring for each $m \in \Omega(C)$. So each factor ring R/Rm, $m \in \Omega(C)$ is a semiprime P.I. ring, the center of which is a field, i.e. the R/Rm are simple Artinian rings. Thus R is biregular. Conversely, if R is biregular then R is regular by Proposition 1.4. □

Next we discuss the case of a left Goldie ring R.

PROPOSITION 1.8. *Let R be a left Goldie ring with center C. The following statements are equivalent*:

(1) *R is a fully left bounded fully left idempotent ring.*

(2) *R is a semisimple Artinian ring.*

(3) *For each $m \in \Omega(C)$, $Q_{C-m}(R)$ is a simple Artinian ring.*

(4) *R is a fully left bounded biregular ring.*

(5) *R is a regular ring.*

(6) *R is a fully left bounded left V-ring.*

PROOF. Except for the implications (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2), all implications are either well-known or they follow in a straightforward way from the theory expounded so far in this paper. Now since an essential left ideal of a semiprime left Goldie ring contains a regular element and since regular elements in a regular ring are invertible, it follows that R contains no proper essential left ideals, hence R is a semisimple Artinian ring, i.e. (5) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2). Consider an essential left ideal I of R which is also two-sided. Since R is a semiprime left Goldie ring, I contains a regular element a. However since R is also fully left idempotent we obtain that: $RaRa = Ra$ whence $RaR = R$, i.e. $I = R$.

Along the lines of the proof of proposition II.7.35 in [4] we establish that R is a finite product of simple rings. Let $M \in \Omega(R)$. If $M = 0$ then R is simple; otherwise there exists a non-zero left ideal L of R such that $L \cap M = 0$ because M cannot be a left essential ideal. Consider $\mathrm{Ann}^r_R(M) = \{r \in R, Mr = 0\}$. Clearly $\mathrm{Ann}^r_R(M)$ is an ideal of R containing L, thus $\mathrm{Ann}^r_R(M) \not\subset M$, and $\mathrm{Ann}^r_R(M) + M = R$. Since R is a semiprime ring $\mathrm{Ann}^r_R(M) \cap M = 0$, hence $R = S \oplus M$, and S is obviously a simple ring. Now M is a semiprime ring such that no proper ideal is essential as a left ideal, thus we may repeat the preceeding decomposition argument. However, R contains no infinite direct sums of non-zero left-ideals of R, thus R has to be a finite product of simple rings, S_i say. As factor rings of R the S_i have to be fully left bounded, i.e. the rings S_i are simple Artinian rings; therefore R is a semisimple Artinian ring.

ODDS AND ENDS. (1) A left Goldie fully left idempotent ring is a finite product of simple rings hence biregular.

(2) An example of B. Osofsky, cf. [12], establishes that a simple Noetherian V-domain need not be Artinian.

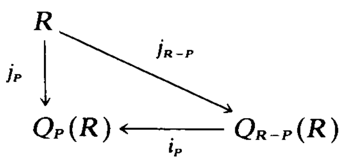
(3) A semiprime, left Artinian ring is a global Zariski extension of its center. An arbitrary Artinian ring need not be a Zariski extension of its center, as may be seen by checking the ring of 2×2 upper triangular matrices over a field.

(4) For a left Noetherian ring R with regular center C it is easily seen that the center of $Q_{C-m}(R) = R/Rm$ is exactly (isomorphic to) C/m, for each $m \in \Omega(C)$. Consequently, a left Noetherian ring is biregular if and only if all central localizations at $m \in \Omega(C)$ turn out to be biregular.

2. Localization at a prime ideal

If R is a global Zariski extension of its center C then symmetric localization at a prime ideal P of R is nothing but central localization at $p = P \cap C$, and thus in this case we always have that Q_{R-P} is an exact functor. On the other hand, localization at the Lambek–Michler kernel functor κ_P (cf. [8]) associated to P yields an exact localization functor Q_P if R satisfies the left Ore conditions with respect to $G(P) = \{r \in R, xr \in P \Rightarrow x \in P\}$. Since, in ivestigating biregularity, fully idempotency, etc. ..., regularity of the center is one of the main facts, it is natural to study the relation between the centers of $Q_P(R)$ and $Q_{R-P}(R)$ respectively.

Recall that, for a global Zariski extension R of its center C and for any prime ideal P of R, we always have that κ_{R-P} is an idempotent kernel functor, even in the absense of the left Noetherian condition, and that $\kappa_{R-P} \leqq \kappa_P$, i.e. $\kappa_{R-P}(M) \subset \kappa_P(M)$ for all $M \in R$-mod. Hence we always have a commutative diagram of canonical ring homomorphisms:



where i_P need not be injective in general.

Proposition 2.1. *Let P be a proper prime ideal of a biregular ring R with center C. Then $\kappa_{R-P}(R) = \kappa_P(R) = P$, i.e. i_P is injective, and the centers of $Q_{R-P}(R)$ and $Q_P(R)$ coincide.*

Proof. By Zariski centrality of R, $\kappa_{R-P}(R) = \kappa_{C-p}(R)$ where $p = P \cap C$. Biregularity of R yields $\kappa_{C-p}(R) = Rp = P$ and P is a maximal ideal of R (take $P \neq 0$). Since $\kappa_P(R) \supset \kappa_{R-P}(R)$ we get $\kappa_P(R) = P$. The second statement is a consequence of the following.

Put $\bar{R} = R/\kappa_{R-P}(R) = R/\kappa_P(R)$ and let E be an injective envelope of $\bar{R}$ in R-mod. Up to identifying rings of quotients with their copies constructed in E we obtain: $\bar{R} \subset Q_{R-P}(R) \subset Q_P(R) \subset E$. The center of $Q_{R-P}(R)$ is the centralisator of $\bar{R}$ in $Q_{R-P}(R)$ hence $Z(Q_{R-P}(R)) \subset Z(Q_P(R))$. Conversely if $q \in Z(Q_P(R))$ then there is a left ideal $L \in \mathscr{L}(\kappa_P)$, the Gabriel filter of κ_P, such that $Lq \subset \bar{R}$. Since q commutes with $\bar{R}$ it also follows that $LRq \subset \bar{R}$. Now LR is an ideal of R not contained in P hence $LR \in \mathscr{L}(R-P)$, cf. [16]. Consequently $q \in Q_{R-P}(R)$ and $q \in Z(Q_{R-P}(R))$ follows. □

NOTE. The second statement may be derived from the following more general lemma: *let R be any ring, P a prime ideal of R such that κ_{R-P} is idempotent, $\kappa_{R-P} \leqq \kappa_P$ and $\kappa_{R-P}(R) = \kappa_P(R)$, then $Q_{R-P}(R)$ and $Q_P(R)$ have the same center (up to isomorphism).*

PROPOSITION 2.2. *If R is a left Noetherian biregular ring then R satisfies the left Ore condition with respect to $G(P)$ for every prime ideal P of R.*

PROOF. Since R/P is a prime left Goldie ring, the set of regular elements of R/P is a left Ore set, i.e. for every $r \in R$, $s \in G(P)$ there exist $r' \in R$, $s' \in G(P)$ such that $s'r - r's \in P$. Since R is biregular we have $P = \kappa_P(R)$, i.e. there exists an $s'' \in G(P)$ such that $s''(s'r - r's) = 0$ we found $s''' \in G(P)$, $r'' \in R$ such that $s'''r - r''s = 0$. □

PROPOSITION 2.3. *Let R be a fully left bounded ring which is a global Zariski extension of its center C, and suppose that C is a regular ring. Then, for each $P \in \operatorname{Spec} R$ we obtain:*

(1) $\kappa_P = \kappa_{R-P}$.

(2) *$G(P)$ is a left reversible Ore set of R.*

PROOF. (1) Let $M \in \Omega(R)$ contain P, then $M \cap C = P \cap C$ follows because C is regular. But then $M = P$ be Zariski centrality of R over C, hence P is maximal. Write $p = P \cap C$ and pick $s \in G(P)$. Since R is a global Zariski extension of C it follows that $\bar{P} = P/Rp$ is the unique prime ideal of $\bar{R} = R/Rp$. Since R is also fully left bounded, $\bar{R}/\bar{P} = R/P$ is a left bounded simple ring hence simple Artinian. Clearly $s \bmod P$ is left regular in $\bar{R}/\bar{P}$, thus invertible in $\bar{R}/\bar{P}$. Now $\bar{P}$ is the Jacobson radical of $\bar{R}$, therefore $s \bmod Rp$ is invertible in $\bar{R}$. The inclusion $\kappa_{R-P} \leqq \kappa_P$ is trivial. Conversely, let $L \in \mathscr{L}(\kappa_p)$, then $Rs \subset L$ for some $s \in G(P)$. The foregoing entails that there is an $s' \in R$ such that $s's - 1 \in Rp$. Moreover $Rp = \kappa_{C-p}(R)$ follows from the fact that C is regular, hence there is a $t \in C - p$ such that $t = ts's$ but this means that $Rs \subset \mathscr{L}(\kappa_{R-P})$ and also that $\kappa_{R-P} = \kappa_P$.

(2) Take $s \in G(P)$. As before we may find $t \in C - p$ such that $Rt \subset Rs$. Then $Rs \in \mathscr{L}(\kappa_P)$ because $t \in G(P)$ too, establishing that $G(P)$ is a left Ore set of R.

Suppose now that $rs = 0$ for some $r \in R$, $s \in G(P)$. As before $s \bmod Rp$ is invertible in R/Rp and from this we derive that $rs = 0$ implies $r \in Rp = \kappa_{C-p}(R)$. Therefore $tr = 0$ for some $t \in C - p$ and this states exactly that $G(P)$ is left reversible. □

REMARK 2.4. Under the conditions of Proposition 2.3 R is also fully right bounded and $G(P)$ coincides with the set $\{g \in R, gr \in P \Rightarrow r \in P\}$. Hence $G(P)$ is also a right reversible right Ore set of R.

PROPOSITION 2.5. *Let R be a ring with regular center C and suppose that R is a finitely generated C-module.*

The following statements are equivalent:

(1) *R is a global Zariski extension of C.*

(2) *$G(P)$ is a left and right Ore set of R for each prime ideal P of R.*

PROOF. (1) $\Rightarrow$ (2). Follows from Proposition 2.3, Remark 2.4.

(2) $\Rightarrow$ (1). For each $q \in \operatorname{Spec} C$ there is a $Q \in \operatorname{Spec} R$ such that $Q \cap C = q$, so we only have to show that $\operatorname{rad} H = \operatorname{rad} R(H \cap C)$ holds for every ideal H of R. If $P \in \operatorname{Spec} R$ write $p = P \cap C$ and let π_p be the canonical epimorphism $R \to R/Rp$. The center of R/Rp is a field, $Z(R/Rp) = \pi_p(C) \cong C/p$ and R/Rp is finitely generated as a C/p-module. From $\pi_p(G(P)) = G(\pi_p(P))$ it follows that $G(\pi_p(P))$ is a left and right Ore set of R/Rp. Then by theorem 6.1 of [10] it follows that $\pi_p(P)$ is the unique prime ideal of R/Rp lying over 0, consequently $\pi_p(P)$ is the unique proper prime ideal of R/Rp. Suppose that H is an ideal of R such that $H \cap C \subset P \cap C$. Evidently, $\pi_p(H)$ is an ideal of R/Rp. If $\pi_p(H) = R/R_p$ then $1 - a \in Rp$ for some $a \in H$ and since $Rp = \kappa_{C-p}(R)$, there exists an $s \in C - p$ such that $s = sa \in H \cap C$, contradiction. Thus $\pi_p(H) \subset \pi_p(P)$, whence $H \subset P$ follows. □

NOTE. If R is finitely generated as a module over its regular center then $\operatorname{Spec} R = \Omega(R)$. In the semiprime case one can say a lot more. Let us summarize this in:

PROPOSITION 2.6. *Let R be a ring with regular center C and suppose that R is finitely generated as a C-module.*

The following statements are equivalent:

(1) *R is semiprime.*

(2) *R is regular.*

(3) *R is biregular.*

(4) *R is a global Zariski extension of C and every prime ideal of R is idempotent.*

(5) *R is an Azumaya algebra over C.*

PROOF. (1) $\Rightarrow$ (2). Cf. [1].

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) have been established before in more general context.

(3) $\Rightarrow$ (5). Take $m \in \Omega(C)$. The center of R/Rm is C/m (up to isomorphism). Since R is biregular, R/Rm is a simple Artinian ring; this yields that R is an Azumaya algebra.

(5) $\Rightarrow$ (1). Observe that C is semiprime and $R(H \cap C) = H$ for any ideal H of R.

(4) $\Rightarrow$ (3). Take $M \in \Omega(R)$ and put $m = M \cap C$. By Zariski centrality of R, M/Rm is the unique prime ideal of the Artinian ring R/Rm. It follows that $M^n \subset R(M \cap C)$ for some $n \in \mathbf{N}$. However, since M is idempotent, $M = R(M \cap C)$ follows. Theorem 1.2 concludes the proof of the fact that R is biregular. □

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